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Analysis on eigenvalues for preconditioning cubic spline collocation method of elliptic equations

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Abstract

In the work of solving a uniformly elliptic differential equations $Au := -\Delta u + a_1 u_x + a_2 u_y + a_0 u = f$ in the unit square with boundary conditions by the C^1 -cubic spline collocation method, one may need to investigate efficient preconditioning techniques. For this purpose, using the generalized field of values argument, we show the uniform bounds of the eigenvalues of the preconditioned matrix when a full finite element preconditioning is considered. © 2001 Elsevier Science Inc. All rights reserved.

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1. Introduction

Let Ω be the unit square $[0, 1] \times [0, 1]$ with its boundary Γ and consider a uniformly elliptic operator A given by

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$$Au := -\Delta u + a_1(x, y)u_x + a_2(x, y)u_y + a(x, y)u \quad \text{in } \Omega, \quad (1.1)$$

where the coefficients $a_1(x, y)$, $a_2(x, y)$ and $a(x, y)$ are smooth functions on Ω . Two types of boundary conditions for (1.1) are given as

$$u = 0 \quad \text{on } \Gamma, \quad (1.2)$$

or

$$\begin{cases} u = 0 & \text{on } \Gamma_D, \\ \frac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \Gamma_N, \end{cases} \quad (1.3)$$

where $\Gamma_D = \{(x, 0) \mid 0 \leq x \leq 1\} \cup \{(0, y) \mid 0 \leq y \leq 1\}$, $\Gamma_N = \Gamma \setminus \Gamma_D$ and $\partial u / \partial \mathbf{n}$ denotes the outward unit normal derivative on Γ_N . Let A_{N^2} be the discretization of the operator A based on the C^1 -cubic spline spaces $S_{h^2,3}$ and the local Legendre–Gauss [=: LG] points, and let \hat{A}_{N^2} be its matrix representation by C^1 -cubic Lagrange spline basis (see Section 2). Recently, there is a report on iterative line spline collocation method in [7] where sharp bounds for spectral radius of the Jacobi iteration matrix are obtained and a spectral analysis is provided for Hermite cubic spline collocation method in [13]. For the orthogonal spline collocation method, fast direct solvers were developed in [1]. Also fast algorithms were reported for high-order spline collocation systems in [12]. By contrast with recent developments of fast direct solver, a preconditioning technique related to the usual finite element method for a polynomial spline collocation method is considered here. The cubic spline collocation method has a property such that the condition number of the matrix \hat{A}_{N^2} increases as a power of $1/h$ ($h = 1/N$). Thus, it is necessary to investigate such condition numbers, which are supposed to be independent of the size of a preconditioned matrix, for the successful applications of the well-known iterative methods such as damped Jacobi iterative method, GMRES, conjugate gradient method, etc.

This paper, which is the continuation of [9], is stimulated by recent work in [10] in which (1.1) is considered with only Dirichlet boundary conditions. For the C^1 -cubic spline collocation method, the uniform bounds of the condition numbers were investigated in [9] or [8], respectively, for the following preconditioned matrices:

$$\beta_{N^2}^{-1} W_{N^2} \hat{A}_{N^2} \quad \text{or} \quad L_{N^2}^{-1} \hat{A}_{N^2}, \quad (1.4)$$

where W_{N^2} is the diagonal matrix of quadrature weights, β_{N^2} and L_{N^2} are the finite element stiffness and finite difference matrices, respectively, corresponding to the uniformly invertible elliptic operator B given by

$$Bv := -\Delta v \quad \text{in } \Omega, \quad (1.5)$$

with the same boundary conditions as A

$$v = 0 \quad \text{on } \Gamma, \quad (1.6)$$

or

$$\begin{cases} v = 0 & \text{on } \Gamma_D, \\ \frac{\partial v}{\partial \mathbf{n}} = 0 & \text{on } \Gamma_N. \end{cases} \quad (1.7)$$

In this work, thanks to Parter, we investigate the eigenvalues of preconditioned matrix

$$\beta_{N^2}^{-1} M_{N^2} \hat{A}_{N^2}, \quad (1.8)$$

where M_{N^2} is the finite element mass matrix corresponding to the operator B with the boundary conditions. The preconditioning matrix $\beta_{N^2}^{-1} M_{N^2}$ for two-dimensional case, which is a full matrix, can be constructed relatively easily by using one-dimensional stiffness and mass matrices. We note that the proposed preconditioned matrix (1.8) is more effective than (1.4) in a point of numerical sense.

One of our main results is to show the following estimates: Assume that $a_1 = a_2 = 0$ in the whole domain. Then for any nonzero complex-valued vector U , there exist two constants A_3 and A_4 , independent of N , such that

$$\operatorname{Re} \left\{ \frac{(W_{N^2} \hat{A}_{N^2} U, U)_{\ell_2}}{(W_{N^2} M_{N^2}^{-1} \beta_{N^2} U, U)_{\ell_2}} \right\} \geq A_3 > 0 \quad (1.9)$$

and

$$\left| \frac{(W_{N^2} \hat{A}_{N^2} U, U)_{\ell_2}}{(W_{N^2} M_{N^2}^{-1} \beta_{N^2} U, U)_{\ell_2}} \right| \leq A_4. \quad (1.10)$$

Of course, these estimates imply the similar estimates for the eigenvalues of our preconditioned matrix $\beta_{N^2}^{-1} M_{N^2} \hat{A}_{N^2}$. For general case, a_1 and a_2 are not identically zero in the whole domain, we mention that the above result can be extended to general β_{N^2} —singular values following the earlier work of Kim and Parter [9]. The multigrid and multilevel methods are studied for quadratic spline collocation method in [5]. In practical implementation of polynomial spline collocation method, one may use a multigrid cycle for β_{N^2} instead of using $\beta_{N^2}^{-1}$ following [3,11].

This paper consists of: in Section 2, we collect some preliminary ideas, notations, etc.; in Section 3, we analyze the eigenvalues of a preconditioned matrix for C^1 -cubic Lagrange spline collocation technique for one-dimensional case; the two-dimensional analysis for eigenvalues, basically developed from one-dimensional argument, is dealt in Section 4.

2. Preliminaries

In this section, we introduce some notations, definitions, and basic facts to be used in the sequel.

Let $I = [0, 1]$ be a unit interval. Let $N > 1$ be an integer and set $h := 1/N$. The ‘knots’ are the points $t_k := kh$, $k = 0, 1, \dots, N$, and $I_k := [t_{k-1}, t_k]$, $k = 1, \dots, N$, is the k th subinterval. Let $P_k(t)$ be the set of all polynomials of degree k or less in t . Let $S_{h,3}$ be the C^1 -cubic spline space as

$$S_{h,3} := \{u \in C^1[0, 1] | u|_{I_k} \in P_3(t), \quad k = 1, \dots, N\}$$

and consider two subspaces $S_{h,3}^d$ and $S_{h,3}^m$ of $S_{h,3}$, which are

$$S_{h,3}^d := \{u \in S_{h,3} \mid u(0) = u(1) = 0\}$$

and

$$S_{h,3}^m := \{u \in S_{h,3} \mid u(0) = u'(1) = 0\}.$$

The collocation points $\{\xi_i\}_{i=1}^{2N}$, which are called the local LG points, are defined by

$$\xi_{2i-1} = t_{i-1} + \frac{h}{2}(1 + \eta_1), \quad \xi_{2i} = t_{i-1} + \frac{h}{2}(1 + \eta_2), \quad i = 1, \dots, N,$$

where $\eta_1 := -(1/\sqrt{3})$ and $\eta_2 := 1/\sqrt{3}$ are the two zeros of the Legendre polynomial of degree 2. For the convenience, let $\xi_0 = 0$ and $\xi_{2N+1} = 1$.

The basis we use for $S_{h,3}$ is the C^1 -cubic Lagrange splines (see [9] for more detail). The basis for $S_{h,3}^d$ or $S_{h,3}^m$ is the functions $\{\psi_i\}_{i=1}^{2N}$ satisfying

$$\psi_i(\xi_k) = \delta_{k,i}, \quad k = 0, \dots, 2N$$

and

$$\psi_i(\xi_{2N+1}) = 0 \quad \text{or} \quad \psi_i'(\xi_{2N+1}) = 0,$$

respectively. Such bases can be constructed using spline tool box in MATLAB package [2] for practical use. For the two-dimensional case, let (N_x, N_y) be any couple of positive integers and set $N^2 := 4N_x N_y$. The local LG points $\{P_\mu\}_{\mu=1}^{N^2}$ in the unit square can be arranged as

$$\{P_\mu\}_{\mu=1}^{N^2} := \{\xi_i\}_{i=1}^{2N_x} \otimes \{\xi_j\}_{j=1}^{2N_y}, \quad \mu = i + 2N_x j.$$

The two-dimensional space $S_{h^2,3}^c := S_{h,3}^c \otimes S_{h,3}^c$, where c denotes d or m , is given as the tensor product of one-dimensional spaces, which has the C^1 -bicubic Lagrange basis functions $\{\Psi_\mu\}_{\mu=1}^{N^2}$, arranged in the same order as LG points,

$$\begin{aligned} \Psi_\mu(x, y) &:= \psi_i(x) \psi_j(y), \quad \mu = i + 2N_x j, \\ i &= 1, \dots, 2N_x, \quad j = 1, \dots, 2N_y. \end{aligned}$$

Define the space $S_{h,1}$ of the piecewise linear functions as

$$S_{h,1} := \{f \in C[0, 1] \mid f|_{[\xi_k, \xi_{k+1}]} \in P_1(t), \quad k = 0, \dots, 2N\},$$

and consider two subspaces $S_{h,1}^d$ and $S_{h,1}^m$ of $S_{h,1}$, which are

$$S_{h,1}^d := \{u \in S_{h,1} \mid f(0) = f(1) = 0\}$$

and

$$S_{h,1}^m := \{u \in S_{h,1} \mid f(0) = f'(1) = 0\}.$$

The basis functions $\{\hat{\phi}_k\}_{k=1}^{2N}$ for $S_{h,1}^d$ or $S_{h,1}^m$ are given by the usual hat functions satisfying

$$\hat{\phi}_k(\xi_l) = \delta_{k,l}, \quad l = 0, 1, \dots, 2N,$$

and

$$\hat{\phi}_k(\xi_{2N+1}) = 0 \quad \text{or} \quad \hat{\phi}'_k(\xi_{2N+1}) = 0,$$

respectively. Similarly, $S_{h^2,1}^c := S_{h,1}^c \otimes S_{h,1}^c$, where c denotes d or m is the two-dimensional space of continuous, piecewise bilinear functions of the form $f(x, y) = a + bx + cy + dxy$ on each subrectangle $[\xi_k, \xi_{k+1}] \times [\xi_l, \xi_{l+1}]$, satisfying proper boundary conditions. The basis functions $\{\hat{\phi}_\mu\}_{\mu=1}^{N^2}$ are given by

$$\begin{aligned} \hat{\phi}_\mu(x, y) &:= \hat{\phi}_k(x)\hat{\phi}_l(y), \quad \mu = k + 2N_x l, \\ k &= 1, \dots, 2N_x, \quad l = 1, \dots, 2N_y. \end{aligned}$$

The usual norm and inner product notations are used. For example, if $U = (u_k)$ and $V = (v_k)$ are K -tuples of complex numbers, then the usual ℓ_2 inner product and ℓ_2 norm are defined as

$$(U, V) := \sum_{k=1}^K u_k \bar{v}_k \quad \text{and} \quad \|U\|^2 = (U, U).$$

3. One-dimensional case

In this section, we consider one-dimensional second-order elliptic boundary value problem given by

$$Au := -u'' + a(t)u = f \quad \text{in } I \tag{3.1}$$

with the homogeneous Dirichlet boundary conditions

$$u(0) = u(1) = 0 \tag{3.2}$$

or the mixed boundary conditions

$$u(0) = u'(1) = 0. \tag{3.3}$$

The C^1 -cubic spline collocation method for (3.1) is as the following: find a C^1 -cubic spline solution $u_N \in S_{h,3}^c$, where c denotes d or m , satisfying

$$Au_N(\xi_i) = -u_N''(\xi_i) + a(\xi_i)u_N(\xi_i) = f(\xi_i), \quad i = 1, \dots, 2N, \tag{3.4}$$

where the collocation points $\{\xi_i\}$ are chosen as the local LG points.

Using the Lagrange basis $\{\psi_i\}$ for $S_{h,3}^c$, the function u_N can be represented as

$$u_N(t) = \sum_{i=1}^{2N} u_i \psi_i(t). \tag{3.5}$$

Then Eqs. (3.4) give rise to the linear system

$$\hat{A}_N U = F, \quad (3.6)$$

where the matrix \hat{A}_N is

$$\hat{A}_N(i, j) = (-\psi_j''(\xi_i) + a(\xi_i)\psi_j(\xi_i))$$

and the vectors U and F are

$$U = (u_1, \dots, u_{2N})^t \quad \text{and} \quad F = (f(\xi_1), \dots, f(\xi_{2N}))^t.$$

Note that the matrix \hat{A}_N is symmetric (see [4,6]) so that $U^* \hat{A}_N U$ is real for any complex vector U .

We now consider the preconditioned matrix

$$\beta_N^{-1} M_N \hat{A}_N, \quad (3.7)$$

where β_N and M_N are the finite element stiffness and mass matrices, respectively, corresponding to the preconditioning operator B ,

$$Bu := -u'' \quad \text{in } I \quad (3.8)$$

with boundary conditions (3.2) or (3.3), using the basis functions $\{\hat{\phi}_i\}$ for $S_{h,1}^c$, hence we have

$$\beta_N(i, j) = \left(\int_I \hat{\phi}_i' \hat{\phi}_j' \, dt \right) \quad \text{and} \quad M_N(i, j) = \left(\int_I \hat{\phi}_i \hat{\phi}_j \, dt \right). \quad (3.9)$$

Consider the generalized field of values

$$\mathcal{F} := \left\{ \frac{(W_N \hat{A}_N U, U)}{(W_N M_N^{-1} \beta_N U, U)} \mid (W_N M_N^{-1} \beta_N U, U) \neq 0 \right\}.$$

For any nonzero complex-valued vector U , it is a well-known fact that the denominator is never zero, and let

$$U = W_N^{-1} M_N V$$

for some complex vector V . Then we have

$$\frac{(W_N \hat{A}_N U, U)}{(W_N M_N^{-1} \beta_N U, U)} = \frac{(W_N \hat{A}_N [W_N^{-1} M_N V], [W_N^{-1} M_N V])}{(\beta_N [W_N^{-1} M_N V], V)}$$

and then the set \mathcal{F} becomes

$$\mathcal{F} = \left\{ \frac{(W_N \hat{A}_N U, U)}{(\beta_N U, V)} \mid U = W_N^{-1} M_N V, \, V \neq 0 \right\}. \quad (3.10)$$

Define an inner product $(X, Y)_h$ for the complex $(K+1)$ -tuples X and Y as the following:

$$(X, Y)_h := \sum_{k=0}^K X_k \bar{Y}_k h_{k+1},$$

where X_k and Y_k are k th elements of the vectors X and Y , respectively.

Since the similar arguments can be applied to the mixed boundary case, we will provide the necessary arguments for the mixed boundary case in the end of this section. We now focus on the homogeneous Dirichlet boundary case, that is, (3.1) and (3.8) have the homogeneous Dirichlet boundary conditions (3.2). Then we want to evaluate the generalized field of values (3.10).

Lemma 3.1. *Let $U = (u_1, \dots, u_{2N})^t$ and $V = (v_1, \dots, v_{2N})^t$ be the nonzero complex-valued vectors such that*

$$U = W_N^{-1} M_N V. \quad (3.11)$$

Then the denominator can be rewritten by using some tridiagonal matrix \mathcal{B} as follows:

$$(\beta_N U, V) = (\mathcal{B} V_\delta, V_\delta)_h, \quad (3.12)$$

where the vector V_δ is given by

$$V_\delta = (v_{\delta,j}) = \left(\frac{v_{j+1} - v_j}{h_{j+1}} \right). \quad (3.13)$$

Proof. The basic property of β_N and simple calculations give

$$(\beta_N U, V) = \sum_{k=0}^{2N} \left[\frac{u_{k+1} - u_k}{h_{k+1}} \right] \left[\frac{\bar{v}_{k+1} - \bar{v}_k}{h_{k+1}} \right] h_{k+1}, \quad (3.14)$$

where, of course, $u_0 = u_{2N+1} = v_0 = v_{2N+1} = 0$.

Since $W_N = \text{diag}(\frac{h}{2}, \dots, \frac{h}{2})$, we get from (3.9)

$$W_N^{-1} M_N = \text{trid}(c_k, d_k, e_k), \quad (3.15)$$

where the elements are

$$\begin{cases} c_k = \frac{h_k}{3h}, & k = 2, \dots, 2N, \\ d_k = \frac{2(h_k + h_{k+1})}{3h}, & k = 1, \dots, 2N, \\ e_k = \frac{h_{k+1}}{3h}, & k = 1, \dots, 2N - 1, \end{cases} \quad (3.16)$$

and $h_k = \xi_k - \xi_{k-1}$ for $k = 1, \dots, 2N + 1$.

For the sake of convenience, let

$$c_1 = \frac{4h_1}{3h}, \quad e_0 = 0, \quad e_{2N} = \frac{4h_{2N+1}}{3h}. \quad (3.17)$$

Then, using $v_0 = v_{2N+1} = 0$, we can rewrite (3.11) as

$$u_k = c_k v_{k-1} + d_k v_k + e_k v_{k+1}, \quad k = 1, \dots, 2N.$$

Hence, we have the different quotients of the vector U in (3.14):

$$\frac{u_1 - u_0}{h_1} = Q_0 \frac{v_1 - v_0}{h_1} + R_0 \frac{v_2 - v_1}{h_2}, \quad (3.18)$$

$$\begin{aligned} \frac{u_{k+1} - u_k}{h_{k+1}} &= L_k \frac{v_k - v_{k-1}}{h_k} + Q_k \frac{v_{k+1} - v_k}{h_{k+1}} \\ &\quad + R_k \frac{v_{k+2} - v_{k+1}}{h_{k+2}} + \rho_k v_k, \quad k = 1, \dots, 2N - 1, \end{aligned} \quad (3.19)$$

$$\frac{u_{2N+1} - u_{2N}}{h_{2N+1}} = L_{2N} \frac{v_{2N} - v_{2N-1}}{h_{2N}} + Q_{2N} \frac{v_{2N+1} - v_{2N}}{h_{2N+1}}, \quad (3.20)$$

where the coefficients L_k , Q_k and R_k are

$$\begin{cases} L_k = c_k \frac{h_k}{h_{k+1}}, & k = 1, \dots, 2N, \\ Q_k = d_{k+1} + (e_{k+1} - e_k), & k = 0, \dots, 2N - 1, \\ Q_{2N} = c_{2N} + d_{2N}, \\ R_k = e_{k+1} \frac{h_{k+2}}{h_{k+1}}, & k = 0, \dots, 2N - 1, \end{cases} \quad (3.21)$$

and the coefficients ρ_k , $k = 1, \dots, 2N - 1$, are

$$\rho_k = \frac{1}{h_{k+1}} ((c_{k+1} - c_k) + (d_{k+1} - d_k) + (e_{k+1} - e_k)). \quad (3.22)$$

Note that (3.16), (3.17) and (3.22) yield

$$\rho_k = 0, \quad k = 1, \dots, 2N - 1,$$

so that (3.14) can be rewritten as, in terms of (3.18)–(3.20) with the convenient notations $L_0 = R_{2N} = 0$,

$$\begin{aligned} (\beta_N U, V) &= \sum_{k=0}^{2N} \left[L_k \frac{v_k - v_{k-1}}{h_k} + Q_k \frac{v_{k+1} - v_k}{h_{k+1}} + R_k \frac{v_{k+2} - v_{k+1}}{h_{k+2}} \right] \\ &\quad \times \left[\frac{\bar{v}_{k+1} - \bar{v}_k}{h_{k+1}} \right] h_{k+1}, \end{aligned}$$

which is (3.12) with the tridiagonal matrix \mathcal{B} defined as

$$\mathcal{B} = \text{trid}(L_k, Q_k, R_k). \quad \square$$

From a simple calculation for the elements L_k , Q_k and R_k of the matrix \mathcal{B} , we have

$$L_k = L_{k+2}, \quad Q_k = Q_{k+2}, \quad R_k = R_{k+2}, \quad k = 1, \dots, 2N - 3,$$

and the explicit representation of the matrix \mathcal{B} as follows;

$$\begin{pmatrix} \frac{1}{3}(1 + \frac{2}{\sqrt{3}}) & \frac{1}{3}(1 + \frac{1}{\sqrt{3}}) & 0 & & & & \\ \frac{1}{3}(\frac{4}{\sqrt{3}} - 2) & \frac{1}{3}(3 - \frac{2}{\sqrt{3}}) & \frac{1}{3}(\frac{4}{\sqrt{3}} - 2) & 0 & & & \\ 0 & \frac{1}{6}(1 + \frac{1}{\sqrt{3}}) & \frac{1}{3}(1 + \frac{2}{\sqrt{3}}) & \frac{1}{6}(1 + \frac{1}{\sqrt{3}}) & 0 & & \\ & 0 & \frac{1}{3}(\frac{4}{\sqrt{3}} - 2) & \frac{1}{3}(3 - \frac{2}{\sqrt{3}}) & \frac{1}{3}(\frac{4}{\sqrt{3}} - 2) & 0 & \\ & & \ddots & \ddots & \ddots & & \\ & & 0 & \frac{1}{3}(\frac{4}{\sqrt{3}} - 2) & \frac{1}{3}(3 - \frac{2}{\sqrt{3}}) & \frac{1}{3}(\frac{4}{\sqrt{3}} - 2) & \\ & & & 0 & \frac{1}{3}(1 + \frac{1}{\sqrt{3}}) & \frac{1}{3}(\frac{2}{\sqrt{3}} + 1) & \end{pmatrix}. \quad (3.23)$$

From Lemma 3.1, we get

$$\operatorname{Re}(\beta_N U, V) = \operatorname{Re}(\mathcal{B}V_\delta, V_\delta)_h, \quad \operatorname{Im}(\beta_N U, V) = \operatorname{Im}(\mathcal{B}V_\delta, V_\delta)_h.$$

In order to investigate the real part of the values of $(\mathcal{B}V_\delta, V_\delta)_h$, consider the symmetric and tridiagonal matrix \mathcal{S} :

$$\mathcal{S} = \frac{1}{2}(\mathcal{B} + \mathcal{B}^T),$$

which is

$$\begin{pmatrix} \frac{1}{3}(1 + \frac{2}{\sqrt{3}}) & \frac{1}{6}(\frac{5}{\sqrt{3}} - 1) & 0 & & & & \\ \frac{1}{6}(\frac{5}{\sqrt{3}} - 1) & \frac{1}{3}(3 - \frac{2}{\sqrt{3}}) & \frac{1}{4}(\sqrt{3} - 1) & 0 & & & \\ 0 & \frac{1}{4}(\sqrt{3} - 1) & \frac{1}{3}(1 + \frac{2}{\sqrt{3}}) & \frac{1}{4}(\sqrt{3} - 1) & 0 & & \\ & 0 & \frac{1}{4}(\sqrt{3} - 1) & \frac{1}{3}(3 - \frac{2}{\sqrt{3}}) & \frac{1}{4}(\sqrt{3} - 1) & 0 & \\ & & \ddots & \ddots & \ddots & & \\ & & 0 & \frac{1}{4}(\sqrt{3} - 1) & \frac{1}{3}(3 - \frac{2}{\sqrt{3}}) & \frac{1}{6}(\frac{5}{\sqrt{3}} - 1) & \\ & & & 0 & \frac{1}{6}(\frac{5}{\sqrt{3}} - 1) & \frac{1}{3}(\frac{2}{\sqrt{3}} + 1) & \end{pmatrix}. \quad (3.24)$$

Hence, we have the uniform bound for the real part of the values of the denominator.

Lemma 3.2. For any nonzero complex-valued vector $V = (v_1, \dots, v_{2N})^t$, we have

$$\operatorname{Re}(\beta_N [W_N^{-1} M_N V], V) \geq 0.1176 (\beta_N V, V)$$

and

$$\operatorname{Re}(\beta_N [W_N^{-1} M_N V], V) \leq 1.1126 (\beta_N V, V).$$

Proof. Gershgorin's theorem applied to the matrix \mathcal{S} gives

$$0.1176 \|V\|^2 \leq \operatorname{Re}(\mathcal{B}V, V) \leq 1.1126 \|V\|^2.$$

Hence,

$$0.1176 (\beta_N V, V) \leq \operatorname{Re}(\mathcal{B}V_\delta, V_\delta)_h \leq 1.1126 (\beta_N V, V). \quad \square$$

We now consider the skew-symmetric and tridiagonal matrix $\tilde{\mathcal{S}}$ for the imaginary part of the values of $(\mathcal{B}V_\delta, V_\delta)_h$,

$$\tilde{\mathcal{S}} = \frac{1}{2}(\mathcal{B} - \mathcal{B}^T)$$

which is

$$\begin{pmatrix} 0 & \frac{1}{2}(1 - \frac{1}{\sqrt{3}}) & 0 & & & \\ \frac{1}{2}(\frac{1}{\sqrt{3}} - 1) & 0 & \frac{1}{12}(\frac{7}{\sqrt{3}} - 5) & 0 & & \\ 0 & \frac{1}{12}(5 - \frac{7}{\sqrt{3}}) & 0 & \frac{1}{12}(5 - \frac{7}{\sqrt{3}}) & 0 & \\ & 0 & \frac{1}{12}(\frac{7}{\sqrt{3}} - 5) & 0 & \frac{1}{12}(\frac{7}{\sqrt{3}} - 5) & 0 \\ & & \ddots & \ddots & \ddots & \\ & & 0 & \frac{1}{12}(\frac{7}{\sqrt{3}} - 5) & 0 & \frac{1}{2}(\frac{1}{\sqrt{3}} - 1) \\ & & & 0 & \frac{1}{2}(1 - \frac{1}{\sqrt{3}}) & 0 \end{pmatrix}. \quad (3.25)$$

Hence, we also have the uniform bound for the imaginary part of the values of the denominator.

Lemma 3.3. For any nonzero complex-valued vector $V = (v_1, \dots, v_{2N})^t$, we have

$$|\operatorname{Im}(\beta_N [W_N^{-1} M_N V], V)| \leq 0.2913 (\beta_N V, V).$$

Proof. The Gershgorin's theorem applied to the matrix $\tilde{\mathcal{S}}$ gives

$$|\operatorname{Im}(\mathcal{B}V, V)| \leq 0.2913 \|V\|^2.$$

Hence,

$$|\operatorname{Im}(\mathcal{B}V_\delta, V_\delta)_h| \leq 0.2913 (\beta_N V, V). \quad \square$$

From Lemmas 3.1–3.3, we have the uniform bound for the values of the denominator.

Theorem 3.4. *For any complex-valued vector $V = (v_1, \dots, v_{2N})^t \neq 0$, there exist positive constants $\mu_1 (\geq 0.1176)$, $\mu_2 (\leq 1.1126)$ and $\mu_3 (\leq 1.1501)$, independent of N , such that*

$$\mu_1(\beta_N V, V) \leq \operatorname{Re}(\beta_N [W_N^{-1} M_N V], V) \leq \mu_2(\beta_N V, V) \quad (3.26)$$

and

$$|(\beta_N [W_N^{-1} M_N V], V)| \leq \mu_3(\beta_N V, V). \quad (3.27)$$

Proof. It follows from Lemmas 3.2 and 3.3. \square

Theorem 3.5. *For any complex-valued vector $V = (v_1, \dots, v_{2N})^t \neq 0$, and $U = W_N^{-1} M_N V$, there exist positive constants μ_4 , μ_5 and μ_6 , independent of N , such that*

$$\mu_4(\beta_N U, U) \leq \operatorname{Re}(\beta_N U, V) \leq \mu_5(\beta_N U, U) \quad (3.28)$$

and

$$|(\beta_N U, V)| \leq \mu_6(\beta_N U, U). \quad (3.29)$$

Proof. From (3.26) and Cauchy–Schwarz inequality, we have

$$\mu_1(\beta_N V, V) \leq \operatorname{Re}(\beta_N U, V) \leq |(\beta_N U, V)| \leq (\beta_N U, U)^{1/2} (\beta_N V, V)^{1/2}.$$

Hence,

$$\mu_1^2(\beta_N V, V) \leq (\beta_N U, U). \quad (3.30)$$

Applying (3.12), then

$$(\beta_N U, U) = (\mathcal{B}V_\delta, \mathcal{B}V_\delta)_h.$$

Since \mathcal{B} is bounded matrix,

$$(\mathcal{B}V_\delta, \mathcal{B}V_\delta)_h \leq 4(V_\delta, V_\delta)_h = 4(\beta_N V, V).$$

Hence,

$$(\beta_N U, U) \leq 4(\beta_N V, V). \quad (3.31)$$

From (3.26), (3.27), (3.30) and (3.31) we have the conclusion. \square

We now have one-dimensional eigenvalue results.

Theorem 3.6. *For any complex-valued vector $U = (u_1, \dots, u_{2N})^t \neq 0$, there exist two positive constants A_1 and A_2 , independent of N , such that*

$$\operatorname{Re} \left\{ \frac{(W_N \hat{A}_N U, U)}{(W_N M_N^{-1} \beta_N U, U)} \right\} \geq A_1 > 0$$

and

$$\left| \frac{(W_N \hat{A}_N U, U)}{(W_N M_N^{-1} \beta_N U, U)} \right| \leq A_2.$$

Moreover, let $\lambda_1, \dots, \lambda_{2N}$ be the eigenvalues of $\beta_N^{-1} M_N \hat{A}_N$. Then for all $k = 1, \dots, 2N$

$$\operatorname{Re}(\lambda_k) \geq A_1 > 0$$

and

$$|\lambda_k| \leq A_2.$$

Proof. Note that there are two positive constants μ_7 and μ_8 (independent of N) such that

$$\mu_7(\beta_N U, U) \leq (W_N \hat{A}_N U, U) \leq \mu_8(\beta_N U, U).$$

Since $(W_N \hat{A}_N U, U)$ is real, using (3.28) and (3.29) we have conclusion.

Moreover, let (λ_k, U_k) be the eigen-pairs of $\beta_N^{-1} M_N \hat{A}_N$. Then

$$\hat{A}_N U_k = \lambda_k M_N^{-1} \beta_N U_k$$

so that

$$\lambda_k = \frac{(W_N \hat{A}_N U_k, U_k)}{(W_N M_N^{-1} \beta_N U_k, U_k)}.$$

Hence, we have the conclusion. \square

Let us focus on the mixed boundary case. Consider the second-order elliptic boundary value problem (3.1) with the mixed boundary conditions (3.3). The preconditioning operator B will be (3.8) with the mixed boundary conditions (3.3).

In Lemma 3.1, the stiffness matrix β_N corresponding to the operator B satisfies

$$(\beta_N U, V) = \sum_{k=0}^{2N-1} \left[\frac{u_{k+1} - u_k}{h_{k+1}} \right] \left[\frac{\bar{v}_{k+1} - \bar{v}_k}{h_{k+1}} \right] h_{k+1}, \quad (3.32)$$

where $u_0 = v_0 = 0$. The elements of matrix (3.15) remain same as (3.16) except

$$d_{2N} = \frac{2(h_k + 3h_{k+1})}{3h},$$

so that, with the convenient constants $c_1 = 4h_1/3h$, $e_0 = e_{2N} = 0$, and $v_0 = 0$, we have the difference quotients (3.18) and (3.19) for $k = 1, \dots, 2N - 2$ and modified (3.20) as

$$\begin{aligned} \frac{u_{2N} - u_{2N-1}}{h_{2N}} &= L_{2N-1} \frac{v_{2N-1} - v_{2N-2}}{h_{2N-1}} \\ &\quad + Q_{2N-1} \frac{v_{2N} - v_{2N-1}}{h_{2N}} + \rho_{2N-1} v_{2N-1} \end{aligned}$$

of the vector U in (3.14). Also we have the same values of the coefficients L_k , Q_k and R_k as (3.21) and the values of ρ_k are

$$\rho_k = 0, \quad k = 1, \dots, 2N - 1.$$

Hence, one may have, with the convenient constants $L_0 = R_{2N-1} = 0$,

$$\begin{aligned} (\beta_N U, V) &= \sum_{k=0}^{2N-1} \left[L_k \frac{v_k - v_{k-1}}{h_k} + Q_k \frac{v_{k+1} - v_k}{h_{k+1}} + R_k \frac{v_{k+2} - v_{k+1}}{h_{k+2}} \right] \\ &\quad \times \left[\frac{\bar{v}_{k+1} - \bar{v}_k}{h_{k+1}} \right] h_{k+1}, \end{aligned}$$

so that Lemma 3.1 holds for the mixed boundary case.

The explicit representation of the matrix \mathcal{B} is the same as (3.23) deleting the last row and column, so that, correspondingly, one can modify the matrix \mathcal{S} and $\tilde{\mathcal{S}}$ deleting the last row and column of matrices in (3.24) and (3.25). Hence, following the same lines of proofs in the above lemmas and theorems, we have the conclusion theorem (Theorem 3.6) for the mixed boundary case.

4. Two-dimensional case

In this section, we consider two-dimensional second-order elliptic operators A and B given by (1.1) and (1.5), respectively. We assume that both A and B have the same boundary conditions (1.2) and (1.6), or (1.3) and (1.7).

Since the Lagrange basis $\{\Psi_\mu\}_{\mu=1}^{N^2}$ is constructed by tensor product of the basis function for one-dimensional space, the C^1 -cubic spline collocation matrix \hat{A}_{N^2} corresponding to the operator A in the space $S_{h^2,3}^c$ with the Lagrange basis $\{\Psi_\mu\}$ is

$$\hat{A}_{N^2} = \hat{A}_{N_x} \otimes W_{N_y} + W_{N_x} \otimes \hat{A}_{N_y}.$$

Also the finite element stiffness matrix β_{N^2} and mass matrix M_{N^2} corresponding to the operator B in the space $S_{h^2,1}^c$ with the basis $\{\hat{\Phi}_\mu\}$ are

$$\beta_{N^2} = M_{N_x} \otimes \beta_{N_y} + \beta_{N_x} \otimes M_{N_y},$$

$$M_{N^2} = M_{N_x} \otimes M_{N_y},$$

and the weight matrix W_{N^2} can be represented as

$$W_{N^2} = W_{N_x} \otimes W_{N_y}.$$

In similar to the one-dimensional case, we consider the generalized field of values for any nonzero complex-valued vector $U = (u_1, \dots, u_{N^2})^t$,

$$\left\{ \frac{(W_{N^2} \hat{A}_{N^2} U, U)}{(W_{N^2} M_{N^2}^{-1} \beta_{N^2} U, U)} \mid (W_{N^2} M_{N^2}^{-1} \beta_{N^2} U, U) \neq 0 \right\}$$

or for any nonzero complex vector $V = (v_1, \dots, v_{N^2})^t$,

$$\left\{ \frac{(W_{N^2} \hat{A}_{N^2} U, U)}{(\beta_{N^2} U, V)} \mid U = W_{N^2}^{-1} M_{N^2} V, V \neq 0 \right\}.$$

First, the matrix in the denominator is

$$\begin{aligned} W_{N^2} M_{N^2}^{-1} \beta_{N^2} &= [W_{N_x} \otimes W_{N_y}] [M_{N_x} \otimes M_{N_y}]^{-1} [M_{N_x} \otimes \beta_{N_y} + \beta_{N_x} \otimes M_{N_y}] \\ &= [W_{N_x} \otimes W_{N_y}] [M_{N_x}^{-1} \otimes M_{N_y}^{-1}] [M_{N_x} \otimes \beta_{N_y} + \beta_{N_x} \otimes M_{N_y}] \\ &= W_{N_x} \otimes W_{N_y} M_{N_y}^{-1} \beta_{N_y} + W_{N_x} M_{N_x}^{-1} \beta_{N_x} \otimes M_{N_y}. \end{aligned}$$

Then from the results of one-dimensional case and [9], we have

$$\begin{aligned} ([W_{N_x} \otimes W_{N_y} M_{N_y}^{-1} \beta_{N_y}] U, U) &= (W_{N_x} U_x, U_x) \otimes ([W_{N_y} M_{N_y}^{-1} \beta_{N_y}] U_y, U_y) \\ &\sim (M_{N_x} U_x, U_x) \otimes (\beta_{N_y} U_y, U_y) \\ &= ([M_{N_x} \otimes \beta_{N_y}] U, U), \end{aligned}$$

and similarly we have

$$([W_{N_x} M_{N_x}^{-1} \beta_{N_x} \otimes M_{N_y}] U, U) \sim ([\beta_{N_x} \otimes W_{N_y}] U, U),$$

where $a \sim b$ means that there exist two positive constants c and C such that

$$c \leq \frac{a}{b} \leq C.$$

Hence, we get the following properties: For any complex-valued vector $V = (v_1, \dots, v_{N^2})^t \neq 0$, and $U = W_{N^2}^{-1} M_{N^2} V$, there exist positive constants μ_4, μ_5 and μ_6 , independent of N , such that

$$\mu_4(\beta_{N^2} U, U) \leq \operatorname{Re}(\beta_{N^2} U, V) \leq \mu_5(\beta_{N^2} U, U)$$

and

$$|(\beta_{N^2} U, V)| \leq \mu_6(\beta_{N^2} U, U).$$

Using the well-known fact that

$$(W_{N^2} \hat{A}_{N^2} U, U) \sim (\beta_{N^2} U, U),$$

we can state the two-dimensional eigenvalue results for the preconditioned matrix $\beta_{N^2}^{-1} M_{N^2} \hat{A}_{N^2}$.

Theorem 4.1. *For a nonzero complex vector $U = (u_1, \dots, u_{N^2})^t$, there are two positive constants A_3 and A_4 , independent of N , such that*

$$\operatorname{Re} \left\{ \frac{(W_{N^2} \hat{A}_{N^2} U, U)}{(W_{N^2} M_{N^2}^{-1} \beta_{N^2} U, U)} \right\} \geq A_3 > 0$$

and

$$\left| \frac{(W_{N^2} \hat{A}_{N^2} U, U)}{(W_{N^2} M_{N^2}^{-1} \beta_{N^2} U, U)} \right| \leq \Lambda_4.$$

Moreover, let $\lambda_1, \lambda_2, \dots, \lambda_{N^2}$ be the eigenvalues of $\beta_{N^2}^{-1} M_{N^2} \hat{A}_{N^2}$. Then for all $k = 1, 2, \dots, N^2$,

$$\operatorname{Re}(\lambda_k) \geq \Lambda_3 > 0$$

and

$$|\lambda_k| \leq \Lambda_4.$$

Remark 4.2. For a general elliptic operator $Au := -\Delta u + a_1 u_x + a_2 u_y + a_0 u$ defined in Ω with a boundary condition, one may develop H^1 -singular values of preconditioning system (1.8) following [10] carefully.

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